Probabilistic Methods in Combinatorics

Instructor: Oliver Janzer

Assignment 8

To solve for the Example class on 15th April. Submit the solution of Problem 1 by Sunday 13th April if you wish feedback on it. Some hints will be given on Friday 11th April.

The solution of each problem should be no longer than one page!

Starred problems are typically harder. Don't worry if you cannot solve them.

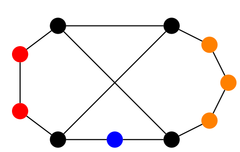
Problem 1. Let n be an integer. Show that, with probability $1 - o(1)$, in $G(n, 1/2)$, all vertices have degree in the range $[n/2 - 2\sqrt{n \log n}, n/2 + 2\sqrt{n \log n}]$.

Problem 2. You are presented with an $n \times n$ grid where each cell in the grid is either red or blue. You can now do the following operation as many times as you like:

Select a row/column and switch the color of each cell in that row/column.

Your goal is to maximize the number of red cells in your grid. Prove that there exists an initial configuration of the grid such that using the operation above arbitrarily many times you cannot turn more than $\frac{1}{2} + \sqrt{\frac{\ln 2}{n}}$ fraction of the cells red.

Problem 3. We say that a graph H is a *subdivision of* K_k (the complete graph on k vertices) if H can be obtained from K_k by replacing each of its $\binom{k}{2}$ edges by inner vertex-disjoint paths (possibly of length 1). For example, the graph below is a subdivision of K_4:



In 1961, György Hajós conjectured that for any $k \in \mathbb{N}$ any graph with chromatic number k contains a subdivision of K_k. This conjecture was disproved by Catlin in 1979, who found counterexamples for $k \geq 7$. The goal of this exercise is to show that with probability $1 - o(1)$ the random graph $G(n, 1/2)$ is a counterexample to Hajós' conjecture.

(a) Show that with probability $1-o(1)$ the random graph $G(n, 1/2)$ has chromatic number at least $n/(10\log_2 n)$.

(b) Show that with probability $1-o(1)$, for every set of $m \geq 100 \ln n$ vertices, out of the $\binom{m}{2}$ possible edges at least $\frac{1}{3}\binom{m}{2}$ are missing.

(c) Use (b) to show that with probability $1-o(1)$ the random graph $G(n, 1/2)$ does not contain a subdivision of K_k for $k \geq 10\sqrt{n}$.

Problem 4*. Prove that the following holds for all large enough n. Let $S_1, \ldots, S_k$ be subsets of $[n] := \{1, \ldots, n\}$. If $k \leq 1.99\frac{n}{\log_2 n}$ then there are two distinct subsets X, Y of $[n]$ such that $|X \cap S_i| = |Y \cap S_i|$ for every $1 \leq i \leq k$.